JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 62 (1995) 321–331

Extensions of Szegő's theory of rational functions orthogonal on the unit circle

K. Pan

Department of Mathematics, Barry University, Miami Shores, FL 33161, USA

Received 5 April 1994; revised 4 October 1994

Abstract

Rational functions orthogonal on the unit circle with prescribed poles lying outside the unit circle are studied. We study the asymptotic behaviors for the orthogonal rational functions when the measure does not satisfy the Szegő condition.

Keywords: Orthogonal functions

1. Introduction

Let $d\mu$ be a finite positive Borel measure with an infinite set as its support on $[0, 2\pi)$. We define $L^2_{d\mu}$ to be the space of all functions $f(z)$ on the unit circle $T := \{z \in \mathbb{C}\colon |z| = 1\}$ satisfying $\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\mu(\theta) < \infty$. Then $L^2_{d\mu}$ is a Hilbert space with inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\mu(\theta).$$

We define $\mathscr{P}_n$ to be all polynomials with degree at most n. For any polynomial q_n with degree n, we define $q_n^*(z) = z^n \overline{q_n(1/\bar{z})}$. Consider an arbitrary infinite triangular array $\boldsymbol{S} = \{z_{n,k}\}$ with $k = 1, \ldots, n$, $n \in \mathbb{N}$ and $|z_{n,k}| < 1$, and let

$$b_{n,k}(z) := \frac{z_{n,k} - z}{1 - \bar{z}_{n,k} z} \frac{|z_{n,k}|}{z_{n,k}}, \quad k = 1, \ldots, n,$$

where for $z_{n,k} = 0$ we put $|z_{n,k}|/z_{n,k} = -1$. Next we define finite Blaschke products recursively as

$$B_{n,0}(z) = 1 \quad \text{and} \quad B_{n,k}(z) = B_{n,k-1}(z) b_{n,k}(z), \quad k = 1, \ldots, n.$$

Elsevier Science B.V.
SSDI 0377-0427(94)00107-3

The fundamental polynomials $w_{n,k}(z)$ are given by

$$w_{n,0}(z) := 1 \quad \text{and} \quad w_{n,k}(z) := \prod_{i=1}^{k} (1 - \bar{z}_{n,k} z), \quad k = 1, \dots, n.$$

The space of rational functions of our interest is defined as

$$\mathscr{R}_{n,m} = \mathscr{R}[z_{n,1}, \dots, z_{n,m}] := \left\{ \frac{p(z)}{w_{n,m}(z)} : p \in \mathscr{P}_m \right\}, \quad n = 0, 1, \dots, \quad m = 0, 1, \dots, n.$$

It is easy to verify that $\{B_{n,k}\}_{k=0}^{m}$ forms a basis of $\mathscr{R}_{n,m}$, i.e., $\mathscr{R}_{n,m} = \operatorname{span}\{B_{n,k}(z), k = 0, \dots, m\}$. Finally, for any $r \in \mathscr{R}_{n,m}$, we define $r^*(z) := B_{n,m}(z)\overline{r(1/\bar{z})}$. Then it is easy to see that $|r^*(z)| = |r(z)|$ for $|z| = 1$ and $r^*(z) \in \mathscr{R}_{n,m}$. For each n, we now define the rational version of Szegő polynomials, orthonormal rational functions, $\phi_{n,m}(z)$, for $m = 0, 1, \dots, n$, $n = 0, 1, 2, \dots,$

$$\phi_{n,m} \in \mathscr{R}_{n,m}, \quad \phi_{n,m}^*(0) > 0,$$

$$\langle \phi_{n,m}, B_{n,k} \rangle = 0, \quad k = 0, \dots, m-1,$$

and

$$\langle \phi_{n,m}, \phi_{n,m} \rangle = 1.$$

The orthogonal rational functions are of constant interest to both mathematicians and physicists. That is because their significance relations between the studies in Hankel and Toeplitz operators, continued fractions, moment problem, Carathéodory–Fejer interpolation, Schur's algorithm and function algebras, and solving electrical engineering problems (cf. [1–9]).

Let $d\mu(\theta) = \mu'(\theta)\,d\theta + d\mu_s(\theta)$ be the Lebesgue decomposition of $d\mu$ with respect to $d\theta$. For the orthogonal rational functions, we proved the following theorem which is similar to the asymptotic behavior for Szegő polynomials:

Theorem A (Pan [16]). *If* $\int_0^{2\pi} \log \mu'(\theta)\,d\theta > -\infty$ *(the Szegő condition) and* $|z_{n,m}| \leqslant r < 1$, $m = 1, \dots, n$, $n = 1, 2, \dots,$ *then*

$$\lim_{n \to \infty} \frac{\phi_{n,n}^*(z)\,(1 - \bar{z}_{n,n} z)}{\sqrt{1 - |z_{n,n}|^2}} = \frac{1}{S(z)}$$

locally uniformly in $z \in D$, *where* $D := \{z : |z| < 1\}$ *and*

$$S(z) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta)\,d\theta \right\}.$$

The goal of the paper is to relax the Szegő condition to the condition $\mu' > 0$, a.e. in $[0, 2\pi)$ and to establish some asymptotic results. The main results are given in Section 2, and their proofs are presented in Section 4. Section 3 is used for citing as well as establishing some auxiliary results that are needed in the proofs of our main results.

2. Main theorems

In this section, we only state our main theorems and the proofs will be given in Section 4. We define the kernel function

$$K_{n,m}(z, w) := \sum_{j=0}^{m} \phi_{n,j}(z) \overline{\phi_{n,j}(w)}.$$

From [1, Theorem 3.1.3], we obtain the following Christoffel–Darboux relation:

$$K_{n,m}(z, w) = \frac{\phi_{n,m}^*(z) \overline{\phi_{n,m}^*(w)} - b_{n,m}(z) \overline{b_{n,m}(w)}\, \phi_{n,m}(z) \overline{\phi_{n,m}(w)}}{1 - b_{n,m}(z)\, \overline{b_{n,m}(w)}}. \tag{2.1}$$

The first result is the relation between $K_{n,m}$ *and* $\phi_{n,m}$. All the proofs of the main theorems depend strongly on the following relation.

Theorem 2.1. *For* $K_{n,m}$ *and* $\phi_{n,m}$, *we have the following*:

$$K_{n,m}(0, 0)\,(z_{n,m} - z)\, \phi_{n,m}(z) = -\,\phi_{n,m}^*(0) z K_{n,m}^*(z, 0) + z_{n,m} \phi_{n,m}(0) K_{n,m}(z, 0).$$

By the above relation, we can prove the following ratio asymptotic behavior.

Theorem 2.2. *If* $\mu' > 0$, *a.e. in* $[0, 2\pi)$ *and* $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$, *then*

$$\lim_{n\to\infty} \frac{\phi_{n,n}^*(0)\, \phi_{n,n-1}^*(z)\, (1 - \bar{z}_{n,n-1} z)}{\phi_{n,n-1}^*(0) \phi_{n,n}^*(z)\ (1 - \bar{z}_{n,n} z)} = 1$$

locally uniformly in $|z| < 1$.

Under a stronger condition for S, we have

Theorem 2.3. *If* $\mu' > 0$, *a.e. in* $[0, 2\pi)$ *and* $|z_{n,m}| \leqslant r < 1$, *then*

$$\lim_{n\to\infty} \frac{\phi_{n,n}^*(z)\ (1 - \bar{z}_{n,n} z)}{\phi_{n,n}(z)\ (z - z_{n,n})} = 0$$

locally uniformly in $|z| > 1$.

Theorem 2.4. *If* $\mu' > 0$, *a.e. in* $[0, 2\pi)$ *and* $|z_{n,m}| \leqslant r < 1$, *then*

$$\lim_{n\to\infty} \frac{\phi_{n,n-1}^*(z)\ (1 - \bar{z}_{n,n-1} z) \sqrt{1 - |z_{n,n-1}|^2}}{\phi_{n,n}^*(z)\ (1 - \bar{z}_{n,n} z) \sqrt{1 - |z_{n,n}|^2}} = 1$$

locally uniformly in $|z| < 1$.

For the norm convergence, we establish the following strong and weak convergence results.

Theorem 2.5. *If $\mu' > 0$, a.e. in $[0, 2\pi)$ and $|z_{n,m}| \leqslant r < 1$, then*

$$\lim_{n\to\infty} \int_0^{2\pi} ||\phi_{n,n}(z)P_n(z)|\sqrt{\mu'(\theta)} - 1|^2 \,\mathrm{d}\theta = 0,$$

where $P_n(z) := (1 - \bar{z}_{n,n} z)/\sqrt{1 - |z_{n,n}|^2}$ and $z = \mathrm{e}^{\mathrm{i}\theta}$.

Theorem 2.6. *If $\mu' > 0$, a.e. in $[0, 2\pi)$ and $|z_{n,m}| \leqslant r < 1$, then, for $z = \mathrm{e}^{\mathrm{i}\theta}$,*

(i) $$\lim_{n\to\infty} \int_0^{2\pi} ||\phi_{n,n}(z)P_n(z)|^2 \mu'(\theta) - 1| \,\mathrm{d}\theta = 0,$$

(ii) $$\lim_{n\to\infty} \int_0^{2\pi} ||\phi_{n,n}(z)P_n(z)|^{-1} - \sqrt{\mu'(\theta)}| \,\mathrm{d}\theta = 0,$$

(iii) $$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\,|\phi_{n,n}(z)P_n(z)|^2 \mu'(\theta) \,\mathrm{d}\theta = \int_0^{2\pi} f(\theta) \,\mathrm{d}\theta$$

for any Borel-measurable function f bounded on $[0, 2\pi)$.

(iv) $$\lim_{n\to\infty} \left(\int_E |\phi_{n,n}(z)\,P_n(z)|^2 \mu'(\theta) \,\mathrm{d}\theta - |E| \right) = 0,$$

holds uniformly as E runs over Borel-measurable subset in $[0, 2\pi)$, where $|E|$ is the Lebesque measure of the set E.

Remark. We still do not know if we can replace the condition $|z_{n,m}| \leqslant r < 1$ by $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$ for the theorems above.

3. Lemmas

In this section we give some lemmas for the needs of the proofs of the theorems. We proved Lemmas 3.1–3.3 in [14, 15] for the special case $z_{n,k} = z_k$. The proofs for the arbitrary infinite triangular array $\boldsymbol{S}$ will be the same as the special case; we leave the proofs to the readers.

Lemma 3.1 (Pan [15]). *If $\mu' > 0$, a.e. in $[0, 2\pi)$ and $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$, then*

$$\lim_{n\to\infty} \frac{K_{n,n-1}(z, 0)}{K_{n,n}(z, 0)} = 1$$

locally uniformly in $|z| < 1$.

Lemma 3.2 (Pan [14]). *If $\mu' > 0$, a.e. in $[0, 2\pi)$ and $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$, then, for $z = \mathrm{e}^{\mathrm{i}\theta}$,*

$$\frac{|\phi_{n,n-k}(0)|^2}{K_{n,n-k}(0, 0)} \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|F_{n,n-k-1}(z)|^2}{|F_{n,n-k}(z)|^2} - 1 \right| \mathrm{d}\theta,$$

and

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|F_{n,n-k-1}(z)|^2}{|F_{n,n-k}(z)|^2}-1\right|\mathrm{d}\theta=0,\quad k=0,1,$$

where

$$F_{n,m}(z):=\frac{K_{n,m}(z,0)}{\sqrt{K_{n,m}(z,0)}}.$$

Thus,

$$\lim_{n\to\infty}\frac{|\phi_{n,n-k}(0)|^2}{K_{n,n-k}(0,0)}=0,\quad k=0,1. \tag{3.1}$$

Lemma 3.3 (Pan [14]). *If $\mu'>0$, a.e. in $[0,2\pi)$ and $\lim_{n\to\infty}\sum_{m=1}^{n}(1-|z_{n,m}|)=+\infty$, then, for $z=\mathrm{e}^{\mathrm{i}\theta}$,*

$$\lim_{n\to\infty}\int_0^{2\pi}||F_{n,n}(z)|\sqrt{\mu'(\theta)}-1|^2\,\mathrm{d}\theta=0.$$

Lemma 3.4. *If $\mu'>0$, a.e. in $[0,2\pi)$ and $|z_{n,m}|\leqslant r<1$, then*

$$\lim_{n\to\infty}\frac{\phi_{n,n}(0)}{\phi_{n,n}^*(0)}=0.$$

Proof. From (2.1), we have

$$K_{n,n}(0,0)=\frac{[\phi_{n,n}^*(0)]^2-|z_{n,n}|^2\,|\phi_{n,n}(0)|^2}{1-|z_{n,n}|^2}.$$

Together with (3.1), we have

$$\lim_{n\to\infty}\frac{|\phi_{n,n}(0)|^2(1-|z_{n,n}|^2)}{[\phi_{n,n}^*(0)]^2-|z_{n,n}|^2\,|\phi_{n,n}(0)|^2}=\lim_{n\to\infty}\frac{|\phi_{n,n}(0)|^2}{K_{n,n}(0,0)}=0. \tag{3.2}$$

Since $|z_{n,m}|\leqslant r<1$, then $(1-|z_{n,n}|^2)/(1-r^2)\geqslant 1$. Also, notice that $|\phi_{n,n}(0)|/|\phi_{n,n}^*(0)|\leqslant 1$, so

$$\frac{1}{1-|z_{n,n}|^2\,|\phi_{n,n}(0)/\phi_{n,n}^*(0)|^2}\geqslant 1.$$

Thus,

$$\lim_{n\to\infty}\frac{|\phi_{n,n}(0)|^2}{[\phi_{n,n}^*(0)]^2}\leqslant\lim_{n\to\infty}\left\{\frac{|\phi_{n,n}(0)|^2}{[\phi_{n,n}^*(0)]^2}\frac{(1-|z_{n,n}|^2)}{(1-r^2)}\frac{1}{(1-|z_{n,n}|^2\,|\phi_{n,n}(0)/\phi_{n,n}^*(0)|^2)}\right\}$$

$$\leqslant \frac{1}{1-r^2} \lim_{n\to\infty} \frac{|\phi_{n,n}(0)/\phi_{n,n}^*(0)|^2 (1-|z_{n,n}|^2)}{1-|z_{n,n}|^2 \, |\phi_{n,n}(0)/\phi_{n,n}^*(0)|^2}$$

$$= \frac{1}{1-r^2} \lim_{n\to\infty} \frac{|\phi_{n,n}(0)|^2 (1-|z_{n,n}|^2)}{[\phi_{n,n}^*(0)]^2 - |z_{n,n}|^2 \, |\phi_{n,n}(0)|^2} = 0.$$

The last equality is from (3.2). □

Recall that $p_{n,m} = \alpha_{n,m} z^m + \cdots, \alpha_{n,m} > 0$, $m = 0, 1, 2, \ldots, n = 1, 2, \ldots,$ is the mth orthonormal polynomial with respect to varying measure $\mathrm{d}\mu/|w_{n,n}(\mathrm{e}^{\mathrm{i}\theta})|^2$.

Lemma 3.5 (Pan [14]). *The following relation holds:*

$$\frac{p_{n,n}^*(z)}{w_{n,n}(z)} = \frac{K_{n,n}(z,0)}{\sqrt{K_{n,n}(0,0)}}.$$

Lemma 3.6 (López [13]). *If* $\mu' > 0$, *a.e. in* $[0, 2\pi)$ *and* $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$, *then*

$$\lim_{n\to\infty} \frac{p_{n,n}^*(z)}{p_{n,n}(z)} = 0,$$

locally uniformly in $|z| > 1$.

Lemma 3.7. *If* $\mu' > 0$, *a.e. in* $[0, 2\pi)$ *and* $\lim_{n\to\infty} \sum_{m=1}^{n} (1 - |z_{n,m}|) = +\infty$, *then*

$$\lim_{n\to\infty} \frac{K_{n,n-k}(z,0)}{K_{n,n-k}^*(z,0)} = 0, \quad k = 0, 1,$$

locally uniformly in $|z| > 1$.

Proof. It is easy to see that the lemma holds from the last two lemmas for $k = 0$.

For $k = 1$, we consider the polynomial $Q_{n-1}(z)$ with degree $n - 1$ which is the $(n-1)$th orthonormal polynomial with respect to $\mathrm{d}\mu/|w_{n,n-1}(\mathrm{e}^{\mathrm{i}\theta})|^2$; then from Lemma 3.5,

$$\frac{Q_{n-1}(z)}{w_{n,n-1}(z)} = \frac{K_{n,n-1}(z,0)}{\sqrt{K_{n,n-1}(0,0)}}. \tag{3.3}$$

From Lemma 3.6, we have

$$\lim_{n\to\infty} \frac{Q_{n-1}^*(z)}{Q_{n-1}(z)} = 0, \tag{3.4}$$

since $\lim_{n\to\infty} \sum_{m=1}^{n-1} (1 - |z_{n,m}|) = \infty$. Together with (3.3) and (3.4), we have

$$\lim_{n\to\infty} \frac{K_{n,n-1}(z,0)}{K_{n,n-1}^*(z,0)} = 0. \qquad \square$$

4. Proofs of main results

Now we are in the position to prove the main results.

Proof of Theorem 2.1. From (2.1), we have

$$K_{n,m}(z,0)=\frac{\phi^*_{n,m}(0)\phi^*_{n,m}(z)-|z_{n,m}|\,\overline{\phi_{n,m}(0)}\phi_{n,m}(z)\,b_{n,m}(z)}{1-|z_{n,m}|b_{n,m}(z)}.$$

Replacing z by $1/\bar{z}$, taking the conjugate and multiplying by $B_{n,m}(z)$ in the above equation, we have

$$K^*_{n,m}(z,0)=\frac{\phi^*_{n,m}(0)\phi_{n,m}(z)-|z_{n,m}|\,\phi_{n,m}(0)\phi^*_{n,m}(z)\,\overline{b_{n,m}(1/\bar{z})}}{1-|z_{n,m}|\overline{b_{n,m}(1/\bar{z})}}.$$

Eliminating $\phi^*_{n,m}(z)$ in the last two identities, notice that $\overline{b_{n,m}(1/\bar{z})}\,b_{n,m}(z)=1$, we have

$$\{[\phi^*_{n,m}(0)]^2-|z_{n,m}|^2\,|\phi_{n,m}(0)|^2\}\,\phi_{n,m}(z)b_{n,m}(z)=\phi^*_{n,m}(0)\,[b_{n,m}(z)-|z_{n,m}|]\,K^*_{n,m}(z,0)$$
$$+\,|z_{n,m}|\,\phi_{n,m}(0)[1-|z_{n,m}|\,b_{n,m}(z)]\,K_{n,m}(z,0).$$

Note that

$$b_{n,m}(z)-|z_{n,m}|=\frac{z(|z_{n,m}|^2-1)}{1-\bar{z}_{n,m}z}\,\frac{|z_{n,m}|}{z_{n,m}},$$

$$1-|z_{n,m}|\,b_{n,m}(z)=\frac{1-|z_{n,m}|^2}{1-\bar{z}_{n,m}z},$$

and

$$K_{n,m}(0,0)=\frac{[\phi^*_{n,m}(0)]^2-|z_{n,m}|^2\,|\phi_{n,m}(0)|^2}{1-|z_{n,m}|^2}.$$

We have

$$K_{n,m}(0,0)\,(z_{n,m}-z)\,\phi_{n,m}(z)=-\,\phi^*_{n,m}(0)\,zK^*_{n,m}(z,0)+z_{n,m}\phi_{n,m}(0)K_{n,m}(z,0). \qquad \square$$

Proof of Theorem 2.2. Using the conjugate form in Theorem 2.1, we have

$$\frac{K_{n,n-1}(0,0)\,(1-\bar{z}_{n,n-1}z)\,\phi^*_{n,n-1}(z)}{K_{n,n}(0,0)\,\,(1-\bar{z}_{n,n}z)\,\phi^*_{n,n}(z)}$$
$$=\frac{-\,\phi^*_{n,n-1}(0)K_{n,n-1}(z,0)+\bar{z}_{n,n-1}\overline{\phi_{n,n-1}(0)}K^*_{n,n-1}(z,0)\,z}{-\,\phi^*_{n,n}(0)K_{n,n}(z,0)+\bar{z}_{n,n}\overline{\phi_{n,n}(0)}K^*_{n,n}(z,0)\,z}$$
$$=\frac{\phi^*_{n,n-1}(0)K_{n,n-1}(z,0)\left[-1+\bar{z}_{n,n-1}\overline{(\phi_{n,n-1}(0)/\phi^*_{n,n-1}(0))}\,(K^*_{n,n-1}(z,0)z/K_{n,n-1}(z,0))\right]}{\phi^*_{n,n}(0)\,K_{n,n}(z,0)\left[-1+\bar{z}_{n,n}\overline{(\phi_{n,n}(0)/\phi^*_{n,n}(0))}\,(K^*_{n,n}(z,0)z/(K_{n,n}(z,0))\right]}. \tag{4.1}$$

Notice that (see, for example [1])

$$\left|\frac{\phi_{n,n+k}(0)}{\phi_{n,n+k}^{*}(0)}\right| \leqslant 1, \quad k=0,-1.$$

From Lemmas 3.7 and 3.1, we have

$$\lim_{n\to\infty}\frac{K_{n,n+k}^{*}(z,0)}{K_{n,n+k}^{*}(z,0)}=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{K_{n,n-1}(z,0)}{K_{n,n}(z,0)}=1$$

locally uniformly in $|z|<1$ for $k=0,-1$. Then we obtain, from (4.1),

$$\lim_{n\to\infty}\frac{\phi_{n,n}^{*}(0)\,(1-\bar{z}_{n,n-1}z)\,\phi_{n,n-1}^{*}(z)}{\phi_{n,n-1}(0)\,(1-\bar{z}_{n,n}z)\,\phi_{n,n}^{*}(z)}=1$$

locally uniformly in $|z|<1$. □

Proof of Theorem 2.3. From Theorem 2.1, we have

$$\begin{aligned}
&\frac{\phi_{n,n}^{*}(z)\,(1-\bar{z}_{n,n}z)}{\phi_{n,n}(z)\,(z-z_{n,n})}\\
&\quad=\frac{-\phi_{n,n}^{*}(0)!!K_{n,n}(z,0)+\bar{z}_{n,n}\overline{\phi_{n,n}(0)}K_{n,n}^{*}(z,0)\,z}{\phi_{n,n}^{*}(0)zK_{n,n}^{*}(z,0)+z_{n,n}\phi_{n,n}(0)K_{n,n}(z,0)}\\
&\quad=\frac{-K_{n,n}(z,0)/K_{n,n}^{*}(z,0)+\bar{z}_{n,n}z\,\overline{(\phi_{n,n}(0)/\phi_{n,n}^{*}(0))}}{z+z_{n,n}\,(\phi_{n,n}(0)/\phi_{n,n}^{*}(0))\,(K_{n,n}(z,0)/K_{n,n}^{*}(z,0))}.
\end{aligned}$$

Thus, together with Lemmas 3.4 and 3.7, we get

$$\lim_{n\to\infty}\frac{\phi_{n,n}^{*}(z)\,(1-\bar{z}_{n,n}z)}{\phi_{n,n}(z)\,(z-z_{n,n})}=0$$

locally uniformly in $|z|>1$. □

Proof of Theorem 2.4. From Theorem 2.2, we only need to prove

$$\lim_{n\to\infty}\frac{\phi_{n,n}^{*}(0)\,(1-|z_{n,n-1}|^{2})}{\phi_{n,n-1}^{*}(0)\,(1-|z_{n,n}|^{2})}=1. \tag{4.2}$$

In fact, from (2.1), we have

$$\begin{aligned}
&\frac{\phi_{n,n}^{*}(0)\,(1-|z_{n,n-1}|^{2})}{\phi_{n,n-1}^{*}(0)\,(1-|z_{n,n}|^{2})}\\
&\quad=\frac{K_{n,n}(0,0)+|\phi_{n,n}(0)|^{2}/(1-|z_{n,n}|^{2})}{K_{n,n-1}(0,0)+|\phi_{n,n-1}(0)|^{2}/(1-|z_{n,n-1}|^{2})}\\
&\quad=\frac{K_{n,n}(0,0)}{K_{n,n-1}(0,0)}\,\frac{[1+|\phi_{n,n}(0)|^{2}/(K_{n,n}(0,0)\,(1-|z_{n,n}|^{2}))]}{[1+|\phi_{n,n-1}(0)|^{2}/(K_{n,n-1}(0,0)\,(1-|z_{n,n-1}|^{2}))]}.
\end{aligned}$$

From Lemmas 3.1 and 3.2, note that $|z_{n,m}| \leqslant r < 1$, we have (4.2). This completes the proof of the theorem. □

Proof of Theorem 2.5. From (2.1), we have

$$F_{n,n}(z) = \frac{K_{n,n}(z,0)}{\sqrt{K_{n,n}(z,0)}} = \frac{\phi_{n,n}^*(0)\,\phi_{n,n}^*(z) - |z_{n,n}|\,\overline{\phi_{n,n}(0)}\,\phi_{n,n}(z)\,b_{n,n}(z)}{[1 - |z_{n,n}|\,b_{n,n}(z)]\sqrt{K_{n,n}(0,0)}}.$$

Notice that

$$K_{n,n}(0,0) = \frac{|\phi_{n,n}(0)|^2 - |z_{n,n}|^2\,|\phi_{n,n}(0)|^2}{1 - |z_{n,n}|^2}$$

and

$$1 - |z_{n,n}|\,b_{n,n}(z) = \frac{1 - |z_{n,n}|^2}{1 - \bar{z}_{n,n}z}.$$

Then

$$\begin{aligned} F_{n,n}(z) &= \frac{\phi_{n,n}^*(0)\,\phi_{n,n}^*(z) - |z_{n,n}|\,\overline{\phi_{n,n}(0)}\,\phi_{n,n}(z)\,b_{n,n}(z)}{\sqrt{|\phi_{n,n}^*(0)|^2 - |z_{n,n}|^2\,|\phi_{n,n}(0)|^2}}\,P_n(z) \\ &= \frac{1 - |z_{n,n}|\,\overline{(\phi_{n,n}(0)/\phi_{n,n}^*(0))}(\phi_{n,n}(z)/\phi_{n,n}^*(z))\,b_{n,n}(z)}{\sqrt{1 - |z_{n,n}|^2\,|\phi_{n,n}(0)|^2/\phi_{n,n}^*(0)}}\,P_n(z)\,\phi_{n,n}^*(z) \\ &=: \frac{N_n(z)}{d_n}\,P_n(z)\,\phi_{n,n}^*(z). \end{aligned}$$

Thus, for $z = \mathrm{e}^{\mathrm{i}\theta}$,

$$\begin{aligned} &\int_0^{2\pi} (|\phi_{n,n}(z)P_n(z)|\sqrt{\mu'(\theta)} - 1)^2\,\mathrm{d}\theta \\ &= \int_0^{2\pi} (|\phi_{n,n}(z)P_n(z)|\sqrt{\mu'(\theta)} - |F_{n,n}(z)|\sqrt{\mu'(\theta)} + |F_{n,n}(z)|\sqrt{\mu'(\theta)} - 1)^2\,\mathrm{d}\theta \\ &\leqslant 2\int_0^{2\pi} \left(|\phi_{n,n}(z)P_n(z)|\sqrt{\mu'(\theta)} - \left|\frac{N_n(z)}{d_n}\,P_n(z)\,\phi_{n,n}^*(z)\right|\sqrt{\mu'(\theta)}\right)^2 \mathrm{d}\theta \\ &\quad + 2\int_0^{2\pi} (|F_{n,n}(z)|\sqrt{\mu'(\theta)} - 1)^2\,\mathrm{d}\theta. \end{aligned}$$

The second term approaches to zero as $n \to \infty$ from Lemma 3.3. Next we prove that the first term approaches to zero. Since, for $z = \mathrm{e}^{\mathrm{i}\theta}$,

$$\begin{aligned} &\int_0^{2\pi} \left(|\phi_{n,n}(z)P_n(z)|\sqrt{\mu'(\theta)} - \left|\frac{N_n(z)}{d_n}\,P_n(z)\,\phi_{n,n}^*(z)\right|\sqrt{\mu'(\theta)}\right)^2 \mathrm{d}\theta \\ &= \int_0^{2\pi} \left(1 - \left|\frac{N_n(z)}{d_n}\right|\right)^2 |P_n(z)\,\phi_{n,n}(z)|^2\,\mu'(\theta)\,\mathrm{d}\theta \\ &\leqslant M_n^2 \int_0^{2\pi} |P_n(z)\,\phi_{n,n}(z)|^2\,\mu'(\theta)\,\mathrm{d}\theta \leqslant M_n^2\,\frac{(1 - |z_{n,n}|)^2}{1 - r^2}, \end{aligned}$$

where

$$M_n := \max_{|z|=1} \left| 1 - \frac{|N_n(z)|}{d_n} \right|.$$

Since $\lim_{n\to\infty} \phi_{n,n}(0)/\phi^*_{n,n}(0) = 0$ and $|\phi_{n,n}(z)/\phi^*_{n,n}(z)| \leqslant 1$, it is easy to see that $\lim_{n\to\infty} M_n = 0$. So, for $z = e^{i\theta}$,

$$\lim_{n\to\infty} \int_0^{2\pi} \left(|\phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} - \frac{|N_n(z)|}{d_n} |P_n(z)\, \phi^*_{n,n}(z)| \sqrt{\mu'(\theta)} \right)^2 d\theta = 0.$$

This completes the proof of the theorem. □

Proof of Theorem 2.6. (i) From Schwarz's inequality, for $z = e^{i\theta}$,

$$\left(\int_0^{2\pi} \| \phi_{n,n}(z) P_n(z)|^2 \mu'(\theta) - 1 | \, d\theta \right)^2$$

$$= \left(\int_0^{2\pi} \| \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} - 1 | \, || \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} + 1 | \, d\theta \right)^2$$

$$\leqslant \int_0^{2\pi} \| \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} - 1 |^2 \, d\theta \cdot \int_0^{2\pi} || \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} + 1 |^2 \, d\theta.$$

The first integral on the last inequality tends to zero as $n \to \infty$ according to Theorem 2.5 and the second one is bounded from Theorem 2.5. Thus (i) follows.

(ii) By Schwarz's inequality, we have

$$\left(\int_0^{2\pi} \| \phi_{n,n}(z) P_n(z)|^{-1} - \sqrt{\mu'(\theta)} | \, d\theta \right)^2$$

$$= \left(\int_0^{2\pi} | \phi_{n,n}(z) P_n(z)|^{-1} \, || \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} - 1 | \, d\theta \right)^2$$

$$\leqslant \left(\int_0^{2\pi} | \phi_{n,n}(z) P_n(z)|^{-2} \, d\theta \right) \left(\int_0^{2\pi} \| \phi_{n,n}(z) P_n(z)| \sqrt{\mu'(\theta)} - 1 |^2 \, d\theta \right).$$

The first integral on the last inequality equals $\int_0^{2\pi} d\mu(\theta)$ (see, for example, [1]) and the second one tends to zero as $n \to \infty$. Thus (ii) follows.

(iii) Is an obvious consequence of (i).

(iv) The uniformity in E of the convergence follows from (i).

This completes the proof of the theorem. □

Acknowledgements

The author thanks the referee for the helpful suggestions.

References

[1] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, A Szegő theory for rational functions, Technical Report TW-131, Dept. of Computer Science, K.U. Leuven, May 1990.

[2] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, The computation of orthogonal rational functions and their interpolating properties, *Numer. Algorithms* **2** (1992) 85–114.

[3] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, A Favard theorem for orthogonal rational functions on the unit circle, *Numer. Algorithms* **3** (1992) 81–89.

[4] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, Moment problems and orthogonal functions, *J. Comput. Appl. Math.* **48** (1993) 49–68.

[5] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, Recurrence relations for orthogonal functions, in: S.C. Cooper and W.J. Thron, Eds., *Continued Fractions and Orthogonal Functions* (Marcel Dekker, New York, 1994) 23–46.

[6] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, Asymptotics for orthogonal rational functions, *Trans. AMS*, to appear.

[7] P. Delsarte and Y. Genin, On the role of orthogonal polynomials on the unit circle in digital signal processing applications, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice* (Kluwer, Dordrecht, 1990) 115–133.

[8] P. Dewilde and H. Dym, Schur recursion, error formulas, and convergence of rational estimators for stationary stochastic sequences, *IEEE Trans. Inform. Theory* **IT27** (1981) 446–461.

[9] M.M. Djrbashian, Orthonormal sets of rational functions on the unit circle with given set of poles, *Doklady SSSR* **147** (1962) 1278–1281 (in Russian).

[10] M.M. Djrbashian, Orthogonal systems of rational functions on the circle, *Izv. Akad. Nauk Armyan. SSR. Ser. Mat.* **1** (1966) 3–24 (in Russian).

[11] M.M. Djrbashian, Orthogonal systems of rational functions on the unit circle, *Izv. Akad. Nauk Armyan. SSR. Ser. Mat.* **1** (1966) 106–125 (in Russian).

[12] M.M. Djrbashian, A survey on the theory of orthogonal systems and some open problems, in: P. Nevai, Ed., *Orthogonal Polynomials: Theory and Practice* (Kluwer, Dordrecht, 1990) 135–146.

[13] G. López, Asymptotics of polynomials orthogonal with respect to varying measures, *Constr. Approx.* **5** (1989) 199–219.

[14] K. Pan, Strong and weak convergence of orthogonal systems of rational functions on the unit circle, *J. Comput. Appl. Math.* **46** (1993) 427–436.

[15] K. Pan, On orthogonal systems of rational functions on the unit circle and polynomials orthogonal with respect to varying measures, *J. Comput. Appl. Math.* **47** (1993) 313–322.

[16] K. Pan, The characterizations for rational functions orthogonal on the unit circle, submitted.